Strategies for managing longevity risk in retirement plans

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SUMMARY

In our work we study the problem of Longevity Risk management and particular interest is given to “revaluating” life annuities. We propose a life annuity model where the payout payment changes dynamically in relation to the fluctuations in the investment return and to the actual mortality trends. The model proposed is effective because, at every annual expiry, the Insurance Company is able to guarantee the life benefit liabilities to an insured collectivity, homogeneous in age and contract model; moreover, it is efficient because, at every annual expiry, also in the presence of unforeseen variations of mortality rates, the Insurance Company has exactly got the mathematical reserve useful to pay annuities to the collectivity members who are still living.

Keywords: Longevity Risk, Observed Probability, Demographic Correction, Rate of Financial Demographical Revaluation, Revaluating Life Annuity under Demographic Compensation (RLADC)

1. **INTRODUCTION**

The evolution of the mortality tendency has shown substantial changes in these recent years that necessarily influence the life insurance market. The most evident phenomena consists in a strong reduction of mortality rates, in particular in the adult and advanced age, and in a progressive shift of the Lexis point, the point in which we observe the maximum death probability in the adult age. These changes are reported in the rectangularization and expansion of the survival curve (see Olivieri and Pitacco, 2002). This reduction in the mortality rates has fundamental and evident consequences on the estimation of the premiums and reserves of all insurance products that involve life benefits, like life annuities. In particular, the actuarial evaluation of the life annuity of an insured party requires the evaluation of the survival function of his own generation. When this evaluation is based on the use of mortality tables of the period, their use is effective until there is no evolution in mortality.

Also the evaluation made with projected mortality tables can be inadequate if the "foreseen" mortality does not agree with the actual successive mortality.

The eventual underestimate of survival probabilities for individuals belonging to a fixed generation involves the risk of an annuity payout for a longer period than expected, and therefore the inadequacy of the annuities total reserve. This risk, deriving from the uncertainty of the future mortality tendency, is well-known in literature as Longevity Risk. There are many different methods to face the Longevity Risk (see Olivieri, 2001); in any case, whatever method is used, the effectiveness or the efficiency can be compromised, if the premium loading request to the annuitants, is respectively too low or too high in comparison to a premium evaluation "a posteriori".

The purpose of this paper consists in the development of an effective and efficient method to manage the Longevity Risk in the case of a life annuity, whose yearly payments are paid out at the end of each year. The method that we suggest is to consider a life annuity that is financially revaluable, in which the unexpected demographical variations are compensated in the financial revaluating "procedure". In particular, it is taken in consideration an insured collectivity, that will be indicated as "G", formed by a group of people, that at the moment of stipulating the contract have already reached the retirement age of x_0 years, without having reached x_0+1 years. The mortality changes for the collectivity G are yearly observed during the annuity payout phase until the total extinction of collectivity G is occurred. Therefore, the Insurance Company can take in account the effect of the mortality evolution during the payout phase and consequently the annuities can be adjusted in a dynamic way which follows the actual, and not foreseen, changes in mortality.

We demonstrate that this method resolves, for the G collectivity, the problem of Longevity Risk in an effective manner, as in every period of the contract the Company is able to payout the annuity, and in an efficient manner, that is, at the end of each year, the Insurance Company has got exactly the mathematical reserve useful to pay annuities to the collectivity members who are still living, also in the presence of unforeseen variations of mortality rates.

We illustrate how this financial demographic technique is feasible and leads to the realization of a new type of life annuity contract that we define as revaluating life annuities under demographical compensation.

This paper is organized as follows. In section 2 a determined collectivity is taken into consideration, "collectivity G," and for this group the survival probabilities and related updating are defined. In section 3 the plan for financial revaluating the life annuity under demographic compensation is defined and the solvency of the Insurance Company is demonstrated.

2. SURVIVAL PROBABILITIES AND UPDATING

Let's say A_0 is a fixed year of the calendar. Let's consider a group of individuals that on Jan 1st of year A_0 have reached age x_0 but haven't yet reached age x_0+1. The members in this group belong to the generation born in the year G, with $G=(A_0-1)-x_0$. These members are indicated as "Collectivity G". Let's suppose that this group is also closed to any new members and that the only reason for withdrawals is death.

We want to go along the mortality evolution of collectivity G until this is exhausted. So, starting from A_0, we have to observe this collectivity for a number of years which is unknown at the beginning of year A_0.

Referring to the members in collectivity G the following survival probabilities are defined.

2.1. <u>*"OBSERVED PROBABILITY" OF SURVIVAL*</u>

Let's *h* denote a natural number. Let's indicate with N_{x_0+h} the number of living members on Jan 1st of year A_0+h, *that we suppose equal to the living members at the end of the year* A_0+h-1. We are aware that a natural number $\bar{h}$ exists such that $N_{x_0+\bar{h}}>0$ and $N_{x_0+\bar{h}+1}=0$, even if we don't know it in advance.

Let's define

Def 1) "*Observed Probability of survival of one year for age* $x_0 + h$" (or simply "*Observed Probability for age* $x_0 + h$") the value given by the formula:

$$\overline{p_{x_0+h}(G)} = \frac{N_{x_0+h+1}}{N_{x_0+h}} \qquad (1)$$

defined for $h \in \mathbb{N}$ such as $h \leq \bar{h}$.

Therefore, the value $\overline{p_{x_0+h}(G)}$ indicates the "Observed Probability" that a member of collectivity G, living on January 1st in year $A_0 + h$, survives one year of calendar. It is evident that the survival probability for age $x_0 + h$ is an "Observed Probability" only at the end of year $A_0 + h$. We have obviously that $\overline{p_{x_0+\bar{h}}(G)} = 0$, seeing that $N_{x_0+\bar{h}+1} = 0$. For convention, if $h > \bar{h}$ we can assume $\overline{p_{x_0+h}(G)} = 0$.

2.2. *"ESTIMATED PROBABILITY" OF SURVIVAL*

Let A denote a generic calendar year such as $A < A_0 + \bar{h}$ and h a natural number. For a member of collectivity G, at the end of year A, if it is $A < A_0 + h$, the probability of one year survival for age $x_0 + h$ is not an observed probability.

For the A and *h* values satisfying the previous condition, Let's define

Def 2) "*Estimated Probability in year A of one year survival for age* $x_0 + h$", (or simply "*Estimated Probability in year A for the age* $x_0 + h$") the value $\tilde{p}^{A}_{x_0+h}$, that gives in calendar year A the probability of one year survival for the member in collectivity G living on January 1st in year $A_0 + h$.

As with the observed probability, let's establish that value $\tilde{p}^{A}_{x_0+h}$ is given only at the end of year A; moreover, this value is, in general, different according to the year the estimate is performed.

"EVALUATED PROBABILITY OF SURVIVAL IN YEAR A"

Let A denote a generic calendar year such as $A < A_0 + \bar{h}$ and h a natural number. From year $A = A_0 + h$, and for each following year, the survival probability of one year for age $x_0 + h$ is an observed probability, equal to $\overline{p_{x_0+h}(G)}$.

For the members in collectivity G, let's define

Def 3) "*Evaluated Probability in year A of one year survival for age* $x_0 + h$" (or simply "*Probability in year A*") the value given by the formula:

$$p^A_{x_0+h}(G) = \begin{cases} \tilde{p}^A_{x_0+h} & \text{if } A < A_0 + h \\ \overline{p_{x_0+h}(G)} & \text{if } A \geq A_0 + h \end{cases} \qquad (2)$$

Therefore, the value $p^A_{x_0+h}(G)$ indicates the "probability in year A" that a living member in collectivity G, on January 1st in year $A_0 + h$, can survive a year and therefore is alive on January 1st the following year. This value is defined whichever calendar year A is; it is given, as established for the observed probability and for the estimated probability, only at the end of year A, and it is equal to an observed or estimated probability depending on the order relation between year A of evaluation and year $A_0 + h$.

In general the following implications apply:

Let G denote the collectivity of individuals which on January 1st in year A_0 *have reached age* x_0*, but not age* $x_0 + 1$*. Let A is any calendar year such as* $A < A_0 + \bar{h}$ *and h a natural number. Therefore we have:*

1) *If in year A the survival probability for age* $x_0 + h$ *is an observed probability, then this probability is an observed probability for any year A' following the year A, that is:*

$$p^A_{x_0+h}(G) = \overline{p_{x_0+h}(G)} \quad \Rightarrow \quad p^{A'}_{x_0+h}(G) = \overline{p_{x_0+h}(G)} \qquad \forall A' / A \leq A' < A_0 + \bar{h} \qquad (3)$$

2) *If in year A the survival probability for age* $x_0 + h$ *is an observed probability, then in year A the survival probability is an observed probability for each age* $x_0 + h'$*, with h' natural number lower or equal to h, that is:*

$$p^A_{x_0+h}(G) = \overline{p_{x_0+h}(G)} \quad \Rightarrow \quad p^A_{x_0+h'}(G) = \overline{p_{x_0+h'}(G)} \qquad \forall h' / 0 \leq h' \leq h \qquad (4)$$

2.3. *"EVALUATED PROBABILITY IN THE YEAR A OF A K YEARS SURVIVAL"*

Let A denote a calendar year such as $A < A_0 + \overline{h}$. Let h indicate a natural number and k a natural number such as $k > 1$. For every value of A, h and k satisfying previous conditions, we can define:

Def 4) "*Evaluated Probability in year A to survive k years starting from age* $x_0 + h$"

(or simply "*Evaluated Probability of k years from age* $x_0 + h$") the value ${}_k p^A_{x_0+h}(G)$ which is given by the formula:

$$ {}_k p^A_{x_0+h}(G) = \prod_{i=0}^{k-1} p^A_{x_0+h+i}(G) \qquad (5) $$

Bear in mind that value ${}_k p^A_{x_0+h}(G)$ indicates the probability evaluated in year A that a member in collectivity G, living on Jan 1st of year $A_0 + h$, survives k years.

Referring to previous definition of Evaluated Probability of one year (see formula (2)), it follows that:

$$ {}_k p^A_{x_0+h}(G) = \begin{cases} \prod_{i=0}^{k-1} \overset{\sim A}{p}_{x_0+h+i} & \text{if } A < A_0 + h \\ \prod_{i=0}^{c_{oss}-1} \overline{p_{x_0+h+i}(G)} \cdot \prod_{i=c_{oss}}^{k-1} \overset{\sim A}{p}_{x_0+h+i} & \text{if } A_0 + h \le A < (A_0 + h) + k - 1 \\ \prod_{i=0}^{k-1} \overline{p_{x_0+h+i}(G)} & \text{if } A \ge (A_0 + h) + k - 1 \end{cases} \qquad (6) $$

where c_{oss} is the useful counter to number the calendar years starting from $A_0 + h$ and therefore to actually count the number of observed probabilities. This is determined by the equation $c_{oss} = A - (A_0 + h) + 1$; when $A = (A_0 + h) + k - 1$ is $c_{oss} = k$, that is , all the one-year survival probabilities are observed.

To simplify notations in the previous (6), we also define:

Def 5) "*Estimated Probability in year A to survive k years starting from age* $x_0 + h$",

(or simply "*Estimated Probability of k years from age* $x_0 + h$") the value ${}_k \overset{\sim A}{p}_{x_0+h}$ given by the formula:

$$ {}_k \overset{\sim A}{p}_{x_0+h} = \prod_{i=0}^{k-1} \overset{\sim A}{p}_{x_0+h+i} \quad \text{with} \quad A < A_0 + h $$

Def 6) “*Observed Probability to survive k years starting from age* $x_0 + h$ ”, (or simply “*Observed Probability of k years from age* $x_0 + h$”) the value $\overline{{}_k p_{x_0+h}(G)}$ given by the formula:

$$\overline{{}_k p_{x_0+h}(G)} = \prod_{i=0}^{k-1} \overline{p_{x_0+h+i}(G)}$$

We note that this value is known only at the end of year $(A_0 + h) + k - 1$. It’s easy to verify that:

$$\overline{{}_k p_{x_0+h}(G)} = \frac{N_{x_0+h+k}}{N_{x_0+h}}$$

Using these definitions, we can express formula (6) in the following form:

$$ {}_k p^A_{x_0+h}(G) = \begin{cases} {}_k \tilde{p}^A_{x_0+h} & \text{if } A < A_0 + h \\ \overline{{}_{c_{oss}} p_{x_0+h}(G)} \cdot {}_{(k-c_{oss})} \tilde{p}^A_{x_0+h+c_{oss}} & \text{if } A_0 + h \le \mathrm{A} < (A_0 + \mathrm{h}) + \mathrm{k} - 1 \\ \overline{{}_k p_{x_0+h}(G)} & \text{if } A \ge (A_0 + h) + k - 1 \end{cases} \qquad (7)$$

Having defined for collectivity G the probabilities evaluated in year A of one year and *k* years survival, for the values of *A* such as $A_0 \le A < A_0 + \overline{h}$ and for each $h \in \mathbb{N}$, it is possible to define

Def 7) “*Expected Number of Complete Years Survival (ENCYS) evaluated in the year A for the age* $x_0 + h$” the value $e^A_{x_0+h}(G)$ given by the following formula:

$$e^A_{x_0+h}(G) = \sum_{k=1}^{+\infty} {}_k p^A_{x_0+h}(G) \qquad (8)$$

We denote with $\tilde{e}^A_{x_0+h}$ the value of ENCYS obtained as the sum of all the estimated probabilities, that is:

$$\tilde{e}^A_{x_0+h} = \sum_{k=1}^{+\infty} {}_k \tilde{p}^A_{x_0+h} \quad \text{with } A < A_0 + h.$$

Let A denote a calendar year such as $A_0 \leq A < A_0 + \overline{h}$. If $A = A_0 + h$, with h natural number, we have $p_{x_0+h}^{A_0+h}(G) = \overline{p_{x_0+h}(G)}$; the other probabilities for the ages following age $x_0 + h$ are all estimated. Therefore, the ENCYS value for age $x_0 + h$ evaluated in year $A_0 + h$ can be expressed by:

$$
\begin{aligned}
e_{x_0+h}^{A_0+h}(G) &= \sum_{k=1}^{+\infty} {}_k p_{x_0+h}^{A_0+h}(G) = \overline{p_{x_0+h}(G)} + \overline{p_{x_0+h}(G)} \cdot \sum_{k=1}^{+\infty} {}_k \tilde{p}_{x_0+h+1}^{A_0+h} = \\
&= \overline{p_{x_0+h}(G)} \cdot \left[1 + \tilde{e}_{x_0+h+1}^{A_0+h} \right]
\end{aligned}
\tag{9}
$$

In general, having i years gone by from year $A_0 + h$, that is, if A is the calendar year such as $A = (A_0 + h) + i$ with i natural number such as $i \geq 1$, for collectivity G the probabilities referred to the ages in the range from $x_0 + h$ to $(x_0 + h) + i$ have been observed, while the probabilities for the ages following $(x_0 + h) + i$ are only estimated. Therefore, the value ENCYS evaluated in year $(A_0 + h) + i$ for age $x_0 + h$ can be expressed as:

$$
\begin{aligned}
e_{x_0+h}^{(A_0+h)+i}(G) &= \sum_{k=1}^{+\infty} {}_k p_{x_0+h}^{(A_0+h)+i}(G) = \\
&= \sum_{k=1}^{i+1} \overline{{}_k p_{x_0+h}(G)} + \overline{{}_{i+1} p_{x_0+h}(G)} \cdot \left(\sum_{k=1}^{+\infty} {}_k \tilde{p}_{(x_0+h)+(i+1)}^{(A_0+h)+i} \right) = \\
&= \sum_{k=1}^{i} \overline{{}_k p_{x_0+h}(G)} + \overline{{}_{i+1} p_{x_0+h}(G)} \cdot \left(1 + \sum_{k=1}^{+\infty} {}_k \tilde{p}_{(x_0+h)+(i+1)}^{(A_0+h)+i} \right) = \\
&= \sum_{k=1}^{i} \overline{{}_k p_{x_0+h}(G)} + \overline{{}_{i+1} p_{x_0+h}(G)} \cdot [1 + \tilde{e}_{(x_0+h)+i+1}^{(A_0+h)+i}]
\end{aligned}
\tag{10}
$$

3. REVALUATING LIFE ANNUITIES UNDER DEMOGRAPHIC COMPENSATION

Let A_0 denote the calendar year at the beginning of which the members of collectivity G have reached age x_0, but haven't yet reached age $x_0 + 1$; x_0 is the age of retirement. Let's suppose that on that date the members of collectivity G stipulate an immediate life annuity contract for a yearly payment, at the end of each year, with the Insurance Company. For this reason, each annuitant pays the Insurance Company a capital M to convert in a life annuity.

Let's establish that, at the beginning of year A_0, the actuarial value of the annuity for age x_0 coincides with the actuarial value of the annuity evaluated at the end of the previous year

$A_0 - 1$. In this case, the actuarial value of life annuity evaluated for age x_0 for a member in collectivity G is defined by the following formula:

$$a_{x_0}^{A_0-1}(G) = \sum_{k=1}^{+\infty} {}_k\, p_{x_0}^{A_0-1}(G) v^k$$

where v^k is the discount factor for k years evaluated at the technical rate. Furthermore, for the sake of simplicity, we suppose that the actuarial evaluation is performed at a technical rate of 0%; then it is:

$$a_{x_0}^{A_0-1}(G) = \tilde{e}_{x_0}^{A_0-1} \qquad (11)$$

where $\tilde{e}_{x_0}^{A_0-1}$ is the ENCYS value which does not contain any observed probability for the collectivity G. Therefore, at the beginning of year A_0, the evaluation of the life annuity payment that we indicate with $R_0^{iA_0}$ is the value that satisfies by the following equation:

$$M = R^{iA_0} \cdot \tilde{e}_{x_0}^{A_0-1} \qquad (12)$$

with $\tilde{e}_{x_0}^{A_0-1} \neq 0$. This payment R^{iA_0} is only “estimated” at the moment of retirement but it is not actually paid out because it is paid at the end of the year and therefore one must consider both the yearly financial revaluation and the updating of the probabilities.

At the end of the first year A_0, the first annuity payment, that is actually paid out by the Insurance Company only to the living members of collectivity G, is equal to the payment calculated at the beginning of the year corrected demographically due to the updating of the survival probabilities and financially revaluated in year A_0.

The first payment of the annuity calculated with the updated survival probabilities at the end of year A_0, that we indicate with R^{A_0}, is given by:

$$R^{A_0} = \frac{M}{e_{x_0}^{A_0}(G)}$$

For (12), we get:

$$R^{A_0} = R^{iA_0} \cdot \frac{\tilde{e}_{x_0}^{A_0-1}}{e_{x_0}^{A_0}(G)} \qquad (13)$$

Therefore, the calculated payment at the end of the year is equal to the payment calculated at the beginning of the same year corrected by a demographic factor that considers, other than the "transformation" of the probability estimated in observed probability for age x_0, also the updating of the probabilities estimated in two different calendar years for ages after x_0.

We define "*Yearly Factor of Demographical Correction (YFDC) in year* A_0 *for age* x_0" the value $cd_{x_0}^{A_0}(G)$ expressed by:

$$cd_{x_0}^{A_0}(G) = \frac{\tilde{e}_{x_0}^{A_0-1}}{e_{x_0}^{A_0}(G)}.$$

The value $cd_{x_0}^{A}(G)$ quantifies the effect of the changes in the dynamics of the mortality of collectivity G and will be less than one if such dynamic results in a general reduction of mortality rates.

Let r^{A_0} denote the yearly interest rate obtained by the Insurance Company and recognized to contract; due to the financial revaluation effect, the first payment of annuity paid out at the end of year A_0, that we indicate as PR^{A_0}, is equal to:

$$PR^{A_0} = \frac{M}{e_{x_0}^{A_0}(G)} \cdot (1 + r^{A_0}) = R^{A_0} \cdot (1 + r^{A_0}) \qquad (14)$$

We also can express the first annuity payment such function of the YFDC; for the (13), we have:

$$PR^{A_0} = R^{iA_0} \cdot cd_{x_0}^{A_0}(G) \cdot (1 + r^{A_0})$$

Regarding the previous equation, we define "*Yearly Rate of Financial Revaluation under Demographical Compensation (YRFRDC) in year* A_0 *for age* x_0", and we indicate it with $r_{x_0}^{A_0}(G)$, the value that satisfies the following equation:

$$1 + r_{x_0}^{A_0}(G) = cd_{x_0}^{A_0}(G) \cdot (1 + r^{A_0})$$

Using this definition, the payment of the annuity paid out to the living members in collectivity G at the end of year A_0 is given by the following equation:

$$PR^{A_0} = R^{iA_0} \cdot (1 + r_{x_0}^{A_0}(G))$$

We see that the previous definitions of the Yearly Factor for the Demographic Correction and Rate of Financial Revaluation under Demographical Compensation can also be extended to the following years of pay out period. In general, whichever is the year $A_0 + h$ and whatever is the age $x_0 + h$, with h natural number such as $h < \overline{h}$, the following are defined:

Def 8) *"Yearly Factor of Demographic Correction YFDC in year $A_0 + h$ for age $x_0 + h$"* is the value $cd_{x_0+h}^{A_0+h}(G)$ expressed by the following:

$$cd_{x_0+h}^{A_0+h}(G) = \frac{\tilde{e}_{x_0+h}^{A_0+h-1}}{e_{x_0+h}^{A_0+h}(G)} \qquad (15)$$

under condition $\tilde{e}_{x_0+h}^{A_0+h-1} \neq 0$.

Def 9) *"Yearly Rate of Financial Revaluation under Demographical Compensation YRFRDC in year $A_0 + h$ for age $x_0 + h$"* is the value $r_{x_0+h}^{A_0+h}(G)$ that satisfies the following equation:

$$1 + r_{x_0+h}^{A_0+h}(G) = cd_{x_0+h}^{A_0+h}(G) \cdot (1 + r^{A_0+h}) \qquad (16)$$

Using the above definitions, it is possible to specify the iterative scheme to calculate the payment of the life annuity. In general, let's establish that at the end of year $A_0 + h$, with h a natural number such as $h < \overline{h}$, the annuity payment is equal to the value paid out at the end of the previous year corrected demographically due to the updating of the survival probability effect starting from age $x_0 + h$ and financially revaluated in year $A_0 + h$. This means that:

$$PR^{A_0+h} = PR^{A_0+h-1} \cdot cd_{x_0+h}^{A_0+h}(G) \cdot (1 + r^{A_0+h}) \qquad (17)$$

where we have supposed that $PR^{A_0-1} = R^{iA_0}$. Equally, referring to the rate of financial demographic revaluation, we establish that:

$$PR^{A_0+h} = PR^{A_0+h-1} \cdot (1 + r_{x_0+h}^{A_0+h}(G)) \qquad (18)$$

with h natural number such as $0 \leq h < \overline{h}$ and $PR^{A_0-1} = R^{iA_0}$.

Let's define

Def 10) "*Revaluating Life Annuities under Demographic Compensation (RLADC)*" is an immediate life annuity whose payments are given out at the end of each year and are expressed by (18).

We want to verify that having used this type of annuity allows the Insurance Company to effectively and efficiently face the Longevity Risk. In particular, the method is effective because, at the end of each year of payout phase, the Insurance Company is able to pay out the annuities matured by the living members in collectivity G. Furthermore the method is efficient because, at the end of each year of payout phase, the Insurance Company, after having paid out annuities to G collectivity members which on that date are still living, has exactly got their mathematical reserves.

Therefore, let N_{x_0} denote the number of members in collectivity G that at the beginning of year A_0 have stipulated a revaluating life annuity contract under demographic compensation with the Insurance Company, and supposing that on this date the members of collectivity G have all paid the same single lump sum M.

In general, $\forall h \in \mathbb{N}$ such as $h < \overline{h}$, we establish that "the end of year $A_0 + h$ goes with the beginning of year $A_0 + h + 1$", and let's indicate with:

1. $N_{x_0+h+1} = N_{x_0+h}\,\overline{p_{x_0+h}(G)}$ the number of members in collectivity G which are still living at the end of year $A_0 + h$;
2. $N_{x_0+h+1} PR^{A_0+h}$ the overall sum paid out by the insurer to pay the life annuities to the members in collectivity G which are still living at the end of year $A_0 + h$;
3. Z_h the fund of the Insurance Company after having paid out life annuities at the end of year $A_0 + h$;
4. $V_h = N_{x_0+h+1} PR^{A_0+h} \tilde{e}_{x_0+h+1}^{A_0+h}$ the technical reserve of revaluating life annuities under demographic compensation for the members in collectivity G which are still living at the end of year $A_0 + h$, that is, at the beginning of year $A_0 + h + 1$.

We want to demonstrate by means of the finite induction principle that, through the payment of a revaluating life annuity under demographic compensation, the Insurance Company is able to meet its obligations at every expiry date, meaning that:

$$Z_h = V_h \qquad (19)$$

for every natural number h being $h < \bar{h}$.

For h=0, at the end of year A_0, the Insurance Company, after having paid out the life annuities to the living members in collectivity G, has a fund Z_0 equal to:

$$Z_0 = N_{x_0} \cdot M \cdot (1 + r^{A_0}) - N_{x_0+1} \cdot PR^{A_0}$$

As a result from definition (1), it is $\overline{p_{x_0}(G)} = \frac{N_{x_0+1}}{N_{x_0}}$, and so substituting in the previous formula, it follows that:

$$Z_0 = \frac{N_{x_0+1}}{\overline{\overline{p_{x_0}(G)}}} \cdot M \cdot (1 + r^{A_0}) - N_{x_0+1} \cdot PR^{A_0}$$

Regarding to the expression (14) that allows expressing M according to the first payment paid out, it is $M(1 + r^{A_0}) = PR^{A_0} e_{x_0}^{A_0}(G)$, and so the previous equation can be re-written in an equivalent manner as:

$$Z_0 = \frac{N_{x_0+1}}{\overline{\overline{p_{x_0}(G)}}} \cdot PR^{A_0} \cdot e_{x_0}^{A_0}(G) - N_{x_0+1} \cdot PR^{A_0} = N_{x_0+1} \cdot PR^{A_0} \cdot \left[\frac{e_{x_0}^{A_0}(G)}{\overline{\overline{p_{x_0}(G)}}} - 1 \right]$$

From formula (9), substituting in the Z_0 expression, it follows that:

$$Z_0 = N_{x_0+1} \cdot PR^{A_0} \cdot \left[\frac{\overline{p_{x_0}(G)} \cdot (1 + \tilde{e}_{x_0+1}^{A_0})}{\overline{\overline{p_{x_0}(G)}}} - 1 \right] = N_{x_0+1} \cdot PR^{A_0} \cdot \tilde{e}_{x_0+1}^{A_0}$$

The quantity $N_{x_0+1} PR^{A_0} \tilde{e}_{x_0+1}^{A_0}$ is the mathematical reserve of the revaluating life annuity under demographic compensation put aside as reserve funds by the Insurance Company at the end of year A_0 to pay out the life annuity performance in the future years to the living members in collectivity G. Therefore, at the end of the first year, it is $Z_0 = V_0$, that is, the fund of the Insurance Company, after having paid the performance of the year, is equal to the mathematical reserve of the revaluating life annuity under demographic compensation for the members in collectivity G.

We demonstrate then that: *For every natural number h such as* $1 \le h < \overline{h}$, *if* $Z_{h-1} = V_{h-1}$ *is true, then* $Z_h = V_h$ *is also true.*

At the end of year $A_0 + h$, the fund of the Insurance Company is equal to:

$$Z_h = Z_{h-1} \cdot (1 + r^{A_0+h}) - N_{x_0+h+1} \cdot PR^{A_0+h}$$

Assuming that, we have $Z_{h-1} = V_{h-1} = N_{x_0+h} \cdot PR^{A_0+h-1} \cdot \tilde{e}_{x_0+h}^{A_0+h-1}$; substituting in the previous equation we see that the Insurance fund is equal to:

$$Z_h = N_{x_0+h} \cdot PR^{A_0+h-1} \cdot \tilde{e}_{x_0+h}^{A_0+h-1} \cdot (1 + r^{A_0+h}) - N_{x_0+h+1} \cdot PR^{A_0+h}$$

By definition (1), it is $\overline{p_{x_0+h}(G)} = \frac{N_{x_0+h+1}}{N_{x_0+h}}$, and from formula (17), it is $PR^{A_0+h} = PR^{A_0+h-1} \frac{\tilde{e}_{x_0+h}^{A_0+h-1}}{e_{x_0+h}^{A_0+h}(G)}(1 + r^{A_0+h})$; so, substituting in the previous equation, we have:

$$Z_h = \frac{N_{x_0+h+1}}{\overline{p_{x_0+h}(G)}} PR^{A_0+h} e_{x_0+h}^{A_0+h}(G) - N_{x_0+h+1} \cdot PR^{A_0+h} =$$

$$= N_{x_0+h+1} PR^{A_0+h} \left(\frac{e_{x_0+h}^{A_0+h}(G)}{\overline{p_{x_0+h}(G)}} - 1 \right)$$

From formula (9), substituting in the Z_h expression, it follows that:

$$Z_h = N_{x_0+h+1} PR^{A_0+h} \left(\frac{\overline{p_{x_0+h}(G)} \cdot (1 + \tilde{e}_{x_0+h+1}^{A_0+h})}{\overline{p_{x_0+h}(G)}} - 1 \right) = N_{x_0+h+1} PR^{A_0+h} \tilde{e}_{x_0+h+1}^{A_0+h} = V_h$$

In this way we have verified that: $Z_{h-1} = V_{h-1} \Rightarrow Z_h = V_h$.

Then, for the finite induction principle, we have $Z_h = V_h$ for every natural number h such as $h < \bar{h}$.

We observe that at the end of the year $A_0 + \bar{h}$ the Insurance Company has a fund expressed by

$$Z_{\bar{h}} = Z_{\bar{h}-1}\left(1 + r^{A_0+\bar{h}}\right)$$

CONCLUSIONS

The majority of the studies carried out on Longevity Risk today present in actuarial literature have focused on the mortality projection problem under different points of view. The contribution of this paper consists in drawing up a life insurance model (RLADC) that allows considering the continuous evolution in the mortality dynamic, financially compensating the unexpected variations of observed rates of mortality. Substantially, in this type of contract the Longevity Risk is quantified and compensated through a financial remuneration rate (yearly rate of financial demographical revaluation) that evolves in a dynamic manner. If in the model there is an efficient scheme of mortality projection the financial compensation of demographical changes can be reduced.

Furthermore with this type of contract, at every expire date, if there are members in collectivity G who are still living, the Insurance Company is able to pay out their life annuities (*effectiveness*) and it has exactly got their reserves (*efficiency*) for the successive years of payout phase.

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